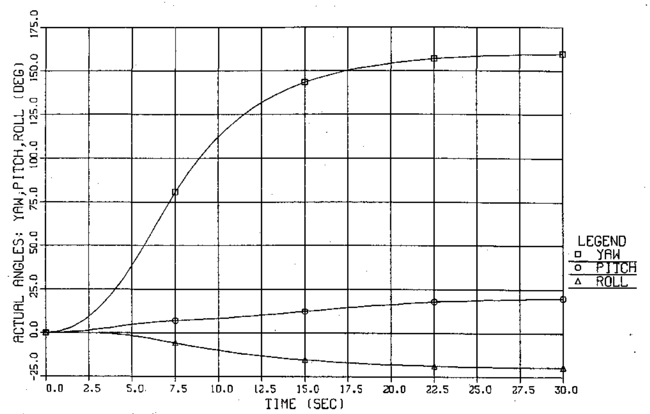

Fig. 1 Three-dimensional control of rigid spacecraft: yaw = 160 deg; pitch = 20 deg; and roll = − 20 deg.

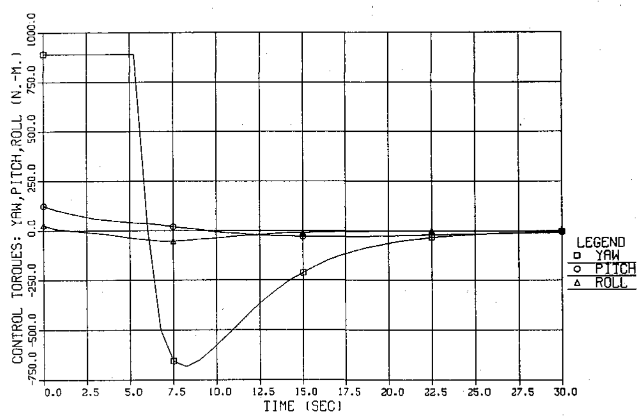

Fig. 2 Three-dimensional slewing control of rigid spacecraft: feedback control torques in yaw, pitch, and roll.

Table 1 Initial guess and converged values of gains and final values of gradient of cost function

Initial k	1.825×10^4	2.464×10^4	1.279×10^4	1.727×10^4	1.819×10^4	2.457×10^4
Final k	1.149×10^4	2.786×10^4	6.334×10^2	3.118×10^3	1.908×10^2	3.169×10^4
$\partial J/\partial k$	-0.82×10^{-4}	0.43×10^{-4}	0.18×10^{-3}	0.12×10^{-5}	0.10×10^{-4}	-0.76×10^{-7}

parameters in Eq. (5): $W_1 = W_2 = W_3 = 2.0 \times 10^3$, $R_1 = 10^{-7}$, $R_2 = R_3 = 10^{-5}$. Figure 1 shows that the desired attitude is captured with little overshoot and with essentially zero terminal angular velocity. The corresponding control torque time histories are given in Fig. 2, showing initial saturation in the yaw torque and the required torque shaping. Table 1 shows the initial guess and final value of the gains, along with the final value of the gradients. It shows that the initial guesses, based on settling time considerations for uncoupled linear equations, were *not* optimal, whereas the gains computed by the procedure given in this paper, while keeping the system stable as indicated by the slewing response, also provided optimal performance.

References

[1]Junkins, J. L., and Turner, J. D., *Optimal Spacecraft Rotational Maneuvers,* Elsevier, Amsterdam, The Netherlands, 1986.

[2]Breakwell, J. A., "Optimal Feedback Control for Flexible Spacecraft," *Journal of Guidance and Control,* Vol. 4, No. 5, 1981, pp. 472–479.

[3]Vadali, S. R., "Feedback Control of Flexible Spacecraft Large Angle Maneuvers Using Liapunov Theory," *Proceedings of the 1984 American Controls Conference,* Inst. of Electrical and Electronics Engineers, Piscataway, NJ, 1984, pp. 1674–1678.

[4]Singh, G., Kabamba, P., and McClamroch, N., "Planar, Time Optimal, Rest-to-Rest Slewing Maneuvers of Flexible Spacecraft," *Journal of Guidance, Control, and Dynamics,* Vol. 12, No. 1, 1989, pp. 71–81.

[5]Wie, B., and Barba, P. M., "Quaternion Feedback for Spacecraft Large Angle Manuevers," *Journal of Guidance, Control, and Dynamics,* Vol. 8, No. 3, 1985, pp. 360–365.

[6]Wertz, J. R. (ed.), *Spacecraft Attitude Determination and Control,* D. Reidel Publishing, 1978, pp. 583, 605.

[7]Bryson, A. E., and Ho, Y.-C., *Applied Optimal Control,* Hemisphere, Washington, DC, 1975.

[8]Hasdorff, L., *Gradient Optimization and Nonlinear Control,* Wiley, New York, 1976.

[9]Van der Plaats, G. N., *Numerical Optimization Techniques for Engineering Design: With Applications,* McGraw-Hill, New York, 1984.

[10]Van der Plaats, G. N., "ADS—A Fortran Program for Automated Design Systhesis," Version 2.01, User's Manual, Engineering Design Optimization, Inc., Santa Barbara, CA, Jan. 1987.

Statistical Linearization for Multi-Input/Multi-Output Nonlinearities

Ching-An Lin*
National Chiao-Tung University, Hsinchu, Taiwan, Republic of China
and
Victor H. L. Cheng†
NASA Ames Research Center, Moffett Field, California 94035

I. Introduction

COVARIANCE analysis provides an alternative to Monte Carlo simulation for evaluating the performance of interconnected nonlinear dynamical systems under noisy environments and, in many practical situations, is more efficient than Monte Carlo simulation.[1,2] Applications of covariance analysis to such nonlinear systems as guidance filters for tactical missiles and space interceptor-target engagement have been reported.[1,3]

A crucial step in the covariance analysis algorithm is to obtain random-input describing functions, i.e., the linear

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*Associate Professor, Department of Control Engineering.
†Research Scientist, Aircraft Guidance and Navigation Branch.

equivalent gains for the nonlinear elements in the system. We call the process of obtaining the optimal linear equivalent gains statistical linearization since the equivalent gains depend on the statistics of the random-input signal. Although statistical linearization for single-input nonlinearities has been studied extensively and equivalent gains for many computed, much less has been done for the general multi-input/multi-output nonlinearities.[4-6]

In this Note, we derive the formulas to compute the random-input describing functions for multi-input/multi-output nonlinearities. The derivations, based on the optimal mean square linear approximation, are straightforward and rigorous. In general, the computations involve evaluations of multiple integrals. We show that, for certain classes of nonlinearities, evaluations of multiple integrals can be avoided and the computations greatly simplified. The formulas developed in this Note have been used in the covariance analysis algorithm in Ref. 3.

II. Statistical Linearization

Typical nonlinearities that we are most concerned with are either memoryless ones as described by

$$y = f_1(u) \tag{1}$$

or dynamical ones described by

$$\dot{x} = f_2(x,u) \tag{2}$$

$$y = f_3(x,u) \tag{3}$$

where u and z are vector random processes with known statistical properties. The objective of statistical linearization is to find linear functions that are the best linear approximations to the nonlinear functions f_1, f_2, and f_3, respectively. Thus, the statistical linearization is formulated for the general nonlinear function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$, where n and m are positive integers.

Let ξ be an n vector of random variables with mean $\bar{\xi} := E\xi$ and covariance matrix $P := E(\xi - \bar{\xi})(\xi - \bar{\xi})^T$. Define $\tilde{\xi} := \xi - \bar{\xi}$ as the zero-mean random part of ξ. The statistical linearization is to find equivalent gain matrices $\bar{N}_f$ and $\tilde{N}_f$, which depend on $\bar{\xi}$ and $\tilde{\xi}$, of size $m \times n$ to approximate the nonlinear function f, so that the mean square of approximation error

$$J(\bar{N}_f, \tilde{N}_f) := E[f(\xi) - \bar{N}_f\bar{\xi} - \tilde{N}_f\tilde{\xi}]^T[f(\xi) - \bar{N}_f\bar{\xi} - \tilde{N}_f\tilde{\xi}] \tag{4}$$

is minimized. The resulting matrices $\bar{N}_f$ and $\tilde{N}_f$ are called the optimal statistical equivalent gains.

For the general state equation

$$\dot{x} = f(x,u)$$

the linearization process yields, as an approximation, linear state equations

$$\dot{\bar{x}} = \bar{N}_f \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \bar{A}\bar{x} + \bar{B}\bar{u}$$

$$\dot{\tilde{x}} = \tilde{N}_f \begin{bmatrix} \tilde{x} \\ \tilde{u} \end{bmatrix} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}$$

where $x = \bar{x} + \tilde{x}$, $u = \bar{u} + \tilde{u}$, $\bar{N}_f = [\bar{A}\ \bar{B}]$, and $\tilde{N}_f = [\tilde{A}\ \tilde{B}]$.

Note that the expected value in Eq. (4) is equal to the sum of the expected values of the m individual square terms, and that each term in the sum depends only on a particular row of matrices $\bar{N}_f$ and $\tilde{N}_f$. Consequently, the minimization problem can be reduced to m simpler ones. Hence, without loss of generality, we consider the case with $m = 1$, and so $\bar{N}_f$ and $\tilde{N}_f$ are $1 \times n$ (row) matrices. For $m = 1$, the mean square approximation error becomes

$$J(\bar{N}_f, \tilde{N}_f) = E[f(\xi) - \bar{N}_f\bar{\xi} - \tilde{N}_f\tilde{\xi}]^2 \tag{5}$$

A necessary condition for $J(\bar{N}_f, \tilde{N}_f)$ to be minimum is

$$D_1 J(\bar{N}_f, \tilde{N}_f) = \theta_{1 \times n} \tag{6}$$

$$D_2 J(\bar{N}_f, \tilde{N}_f) = \theta_{1 \times n} \tag{7}$$

where $D_i J(\bar{N}_f, \tilde{N}_f)$ is the derivative of J with respect to the ith variable evaluated at $(\bar{N}_f, \tilde{N}_f)$, and θ represents a zero matrix of appropriate dimension.

From Eq. (6), the optimal $\bar{N}_f$ must satisfy

$$2\bar{N}_f\bar{\xi}\bar{\xi}^T - 2Ef(\xi)\bar{\xi}^T = \theta_{1 \times n} \tag{8}$$

Clearly, if $\bar{N}_f$ satisfies

$$\bar{\xi}^T \bar{N}_f^T = Ef(\xi) \tag{9}$$

then it satisfies Eq. (8). We shall solve Eq. (9), which is easier.

The solution of Eq. (9) exists and is in general not unique, since the map defined by $\bar{\xi}^T : \bar{N}_f^T \mapsto \bar{\xi}^T \bar{N}_f^T$ is a map of $\mathbb{R}^n \rightarrow \mathbb{R}$ and hence is in general not 1-1. The minimum-norm solution of Eq. (9) can be constructed, for $\bar{\xi} \neq \theta_{n \times 1}$, as in Ref. 7:

$$\bar{N}_f = \frac{\bar{\xi}^T Ef(\xi)}{\bar{\xi}^T \bar{\xi}} \tag{10}$$

Similarly, from Eq. (7), the optimal $\tilde{N}_f$ must satisfy

$$2\tilde{N}_f E(\tilde{\xi}\tilde{\xi}^T) - 2E[\tilde{\xi} f(\xi)] = 0 \tag{11}$$

Now since the covariance matrix $P := E(\tilde{\xi}\tilde{\xi}^T)$ is positive definite and hence nonsingular, the unique solution is given by

$$\tilde{N}_f = E[\tilde{\xi} f(\xi)] P^{-1} \tag{12}$$

Thus, Eqs. (10) and (12) are the formulas to compute the random-input describing functions. For multiple-output nonlinearities each output is considered separately. Note that the optimal equivalent gains are a function of the mean $\bar{\xi}$ and the covariance matrix P.

It is clear from Eqs. (10) and (12) that, to compute $\bar{N}_f$ and $\tilde{N}_f$, we have to evaluate the expected values of $Ef(\xi)$ and $E[\tilde{\xi} f(\xi)]$. Computing the expected values $Ef(\xi)$ and $E[\tilde{\xi} f(\xi)]$ requires the knowledge of the probability distribution of the random variable ξ. In computations, it is usually assumed that the probability density function is known.

III. Computational Aspects

We assume that the random vector under consideration is jointly Gaussian with mean $\bar{\xi} := E(\xi)$ and covariance matrix $P := E(\xi - \bar{\xi})(\xi - \bar{\xi})^T$. More precisely, the probability density function of ξ is

$$p_{\Xi}(\xi) = \{1/[(2\pi)^{n/2}(\det P)^{1/2}]\} \exp[-\tfrac{1}{2}(\xi - \bar{\xi})^T P^{-1}(\xi - \bar{\xi})] \tag{13}$$

We shall first discuss the computations involved in the evaluation of $\bar{N}_f$ and $\tilde{N}_f$ for the single variable case. For the scalar case, $\bar{N}_f$ and $\tilde{N}_f$ in Eqs. (10) and (12) can be computed by

$$\bar{N}_f = \frac{1}{\sqrt{2\pi}\sigma\bar{\xi}} \int_{-\infty}^{\infty} f(\xi) \exp\left[-\frac{(\xi - \bar{\xi})^2}{2\sigma^2} d\xi\right] \tag{14}$$

$$\tilde{N}_f = \frac{1}{\sqrt{2\pi\sigma^3}} \int_{-\infty}^{\infty} (\xi - \bar{\xi}) f(\xi) \exp\left[-\frac{(\xi - \bar{\xi})^2}{2\sigma^2} d\xi\right] \tag{15}$$

where σ^2 is the variance of ξ. Note that the equivalent gain $\bar{N}_f$ and $\tilde{N}_f$ depends on the values of mean $\bar{\xi}$ and variance σ^2. In general, $\bar{N}_f$ and $\tilde{N}_f$, as functions of $\bar{\xi}$ and σ^2, do not have closed-form expressions. From a computational point of view, the integrations in Eqs. (14) and (15) should be carried out off-line for a number of different values of $\bar{\xi}$ and σ^2, and approximate curve-fitting algorithms can be used to obtain a closed-form approximation of $\bar{N}_f$ and $\tilde{N}_f$.

For the multivariable case,

$$\bar{N}_f = \frac{1}{\bar{\xi}^T\bar{\xi}}\,\bar{\xi}^T Ef(\xi)$$

$$= \frac{1}{\bar{\xi}^T\bar{\xi}}\,\bar{\xi}^T\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\xi)p_{\Xi}(\xi)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_n \tag{16}$$

and

$$\tilde{N}_f = \left[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(\xi-\bar{\xi})^T f(\xi)p_{\Xi}(\xi)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_n\right]P^{-1} \tag{17}$$

where P is defined in Eq. (13). Note that Eqs. (16) and (17) involve multiple integrals that are difficult to evaluate numerically. For some special cases, the multiple integral can be reduced to simple forms as discussed in the next section.

IV. Special Cases

A. $f(\xi)$ as a Multivariable Polynomial in the Components of $\xi := [\xi_1\ \xi_2 \ldots \xi_n]^T$

Since the covariance matrix P is positive definite, we can write P in its Cholesky decomposition form $P = RR^T$, where R is lower triangular.[7] Let

$$z := R^{-1}\xi \quad \bar{z} := R^{-1}\bar{\xi}$$

It can be shown that[8]

$$p_Z(z) = p_{\Xi}(Rz)\det(R)$$

$$= [1/(2\pi)^{n/2}(\det P)^{1/2}]$$

$$\times \exp[-\tfrac{1}{2}(Rz-R\bar{z})^T(RR^T)^{-1}(Rz-R\bar{z})]\,(\det R)$$

$$= [1/(2\pi)^{n/2}]\exp[-\tfrac{1}{2}(z-\bar{z})^T(z-\bar{z})]$$

$$= [1/(2\pi)^{n/2}]\exp[-\tfrac{1}{2}\textstyle\sum_{k=1}^{n}(z_k-\bar{z}_k)^2] \tag{18}$$

where we have used the relationship $\det R = (\det P)^{1/2}$ and $(RR^T)^{-1} = R^{-T}R^{-1}$.

Equation (18) shows that the new random vector z is Gaussian, and its covariance matrix is the identity matrix. Define the function $F: \mathbb{R}^n \to \mathbb{R}$ by

$$F(z) := f(Rz) \tag{19}$$

Since f is a polynomial in the components of ξ, F is also a polynomial in the components of $z = [z_1\ z_2 \ldots z_n]^T$. Using Eqs. (18) and (19), we have

$$Ef(\xi) = EF(z) = \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}F(z)$$

$$\times\exp[-\tfrac{1}{2}\textstyle\sum_{k=1}^{n}(z_k-\bar{z}_k)^2]\,\mathrm{d}z_1\cdots\mathrm{d}z_n \tag{20}$$

Since $F(z)$ is a polynomial in the components of z, the multiple integral, Eq. (20), can be decomposed as a sum of products of simple integrals. The computation of $\bar{N}_f$ in Eq. (10) is thus greatly simplified. Similarly, $\tilde{N}_f$ can be computed by

$$\tilde{N}_f = \left\{\frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}[R(z-\bar{z})]^T F(z)\right.$$

$$\left.\times\exp[-\tfrac{1}{2}\textstyle\sum_{k=1}^{n}(z_k-\bar{z}_k)^2]\,\mathrm{d}z_1\cdots\mathrm{d}z_n\right\}P^{-1} \tag{21}$$

$$= \text{sum of products of simple integrals} \tag{22}$$

A typical example of this class of nonlinear functions is the moment equation

$$\dot{p} = -qr(I_z - I_y)/I_x + M_x/I_x \tag{23}$$

$$\dot{q} = -pr(I_x - I_z)/I_y + M_y/I_y \tag{24}$$

$$\dot{r} = -pq(I_y - I_x)/I_z + M_z/I_z \tag{25}$$

where $[p\ q\ r]^T$, $[M_x\ M_y\ M_z]^T$, $[I_x\ I_y\ I_z]^T$ are, respectively, vectors of the angular rate, the moment, and the moment of inertia.

B. $f: \mathbb{R}^2 \to \mathbb{R}$ Defined by $f(\xi_1, \xi_2) = \xi_1 g(\xi_2)$, Where g Is a Scalar Function[6]

With the same definitions as in the previous case, namely Eq. (13), write

$$P = \begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}$$

for some $|\rho| < 1$. It can be verified that the matrix

$$R = \begin{bmatrix}\sigma_1\sqrt{1-\rho^2} & \rho\sigma_1\\ 0 & \sigma_2\end{bmatrix}$$

satisfies that $RR^T = P$. From Eq. (18), the probability density function of $z := R^{-1}\xi$ is

$$p_Z(z) = (1/2\pi)\exp\{-\tfrac{1}{2}[(z_1-\bar{z}_1)^2 + (z_2-\bar{z}_2)^2]\} \tag{26}$$

The expected value

$$Ef(\xi_1,\xi_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\xi_1 g(\xi_2)p_{\Xi}(\xi)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\sigma_1\sqrt{1-\rho^2}z_1 + \sigma_1\rho z_2)g(\sigma_2 z_2)p_Z(z)\,\mathrm{d}z_1\,\mathrm{d}z_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\sigma_1\sqrt{1-\rho^2}z_1 + \sigma_1\rho z_2)g(\sigma_2 z_2)\frac{1}{2\pi}$$

$$\times\exp\{-\tfrac{1}{2}[(z_1-\bar{z}_1)^2 + (z_2-\bar{z}_2)^2]\}\,\mathrm{d}z_1\,\mathrm{d}z_2$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\sigma_1\sqrt{1-\rho^2}\bar{z}_1 g(\sigma_2 z_2) + \sigma_1\rho z_2 g(\sigma_2 z_2)]$$

$$\times\exp[-\tfrac{1}{2}(z_2-\bar{z}_2)^2]\,\mathrm{d}z_2 \tag{27}$$

Note that we have used

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}z_1\exp[-\tfrac{1}{2}(z_1-\bar{z}_1)^2]\,\mathrm{d}z_1 = \bar{z}_1$$

and

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp[-\tfrac{1}{2}(z_1-\bar{z}_1)^2]\,\mathrm{d}z_1 = 1$$

Similarly

$$\tilde{N}_f = \left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\xi_1-\bar{\xi}_1\ \ \xi_2-\bar{\xi}_2]\xi_1 g(\xi_2)p_{\Xi}(\xi)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\right\}P^{-1}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\sigma_1\sqrt{1-\rho^2}\,(z_1-\bar{z}_1)$$

$$+ \sigma_1\rho(z_2-\bar{z}_2)\ \ \sigma_2(z_2-\bar{z}_2)]$$

$$\times(\sigma - 1\sqrt{1-\rho^2}z_1 + \sigma_1\rho z_2)g(\sigma_2 z_2)$$

$$\times\exp\{-\tfrac{1}{2}[(z_1-\bar{z}_1)^2 + (z_2-\bar{z}_2)^2]\}\,\mathrm{d}z_1\,\mathrm{d}z_2\,P^{-1} \tag{28}$$

Thus,

$$\tilde{N}_f = \begin{bmatrix} -0.6E(z_1^3) - 0.8E(z_1^2)\bar{z}_2 + \bar{z}_3\bar{z}_1 - 0.6E(z_1^2) + 0.8\bar{z}_1\bar{z}_2 - \bar{z}_3 \\ -0.36E(z_1^3) + 0.6E(z_1^2)0.96E(z_1^2)\bar{z}_2 + 0.8\bar{z}_1\bar{z}_2 - \bar{z}_3 + 0.6\bar{z}_1\bar{z}_3 - 0.64\bar{z}_1E(z_1^2) + 0.8\bar{z}_2\bar{z}_3 \\ -0.6E(z_1^2)\bar{z}_3 - 0.8\bar{z}_1\bar{z}_2\bar{z}_3 + E(z_3^2) + 0.6E(z_1^2) + 0.8\bar{z}_1\bar{z}_2 - \bar{z}_3 \end{bmatrix}^T P^{-1}$$

$$= [-1.0 \quad -1.0 \quad 1.0]$$

Equation (28) can again be reduced to a simple integral by first integrating with respect to dz_1 as in Eq. (27).

C. $f(\xi) = L(\xi_1, \ldots, \xi_{n-1})g(\xi_n)$, Where $\xi = [\xi_1^T \ldots \xi_n^T]^T$, L Is a Multivariable Polynomial in $\xi_1, \ldots, \xi_{n-1}$ and g Is a Scalar Nonlinear Function

Here we decompose P as $P = RR^T$ with R upper triangular. Note that such decomposition can be obtained by performing the Cholesky decomposition on P^{-1}.[7] Let $z = R^{-1}\xi$ and the remaining manipulations are the same as in Secs. IVA and IVB.

V. Illustrative Example

Let us consider the three-input nonlinearity described by

$$F(x) = -x_1x_2 + x_3 \tag{29}$$

where $x = [x_1\ x_2\ x_3]^T$. Note that Eq. (29) has the same form as the right side of the moment Eqs. (23-25). Assume that the random vector x is jointly Gaussian with mean $\bar{x} := Ex = [1\ \ 1\ \ 1]^T$ and covariance matrix

$$P := E(x-\bar{x})(x-\bar{x})^T = \begin{bmatrix} 1 & 0.6 & 0 \\ 0.6 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

We wish to find $\bar{N}_f \in \mathbb{R}^{1\times 3}$ and $\tilde{N}_f \in \mathbb{R}^{1\times 3}$ such that the linear function $L(x) = \bar{N}_f\bar{x} + \tilde{N}_f(x-\bar{x})$ is the best linear approximation to $f(x)$.

To compute $\bar{N}_f$ and $\tilde{N}_f$ by Eq. (16) and Eqs. (18-21), first compute the Cholesky factor R of P to get

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and let $z := R^{-1}x$, $\bar{z} := R^{-1}\bar{x}$. The random variable $z = [z_1\ z_2\ z_3]^T$ is jointly Gaussian with covariance matrix equal to identity and $\bar{z} = Ez = [1\ \ 0.5\ \ 1]$. Let $f(z) := F(Rz) = -0.6z_1^2 - 0.8z_1z_2 + z_3$. We have

$$EF(z) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (0.6z_1^2 - 0.8z_1z_2 + z_3) \times \exp[-\tfrac{1}{2}\textstyle\sum_{k=1}^{3}(z_k - \bar{z}_k)^2]\, dz_1\, dz_2\, dz_3 \tag{30}$$

$$= 0.6E(z_1^2) - 0.8\bar{z}_1\bar{z}_2 + \bar{z}_3 = 1.8$$

where

$$E(z_1^2) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} z_1^2 \exp[-(z_1 - \bar{z}_1)^2]\, dz_1$$

Note that the multiple integral in Eq. (30) can be decomposed as a sum of products of simple integrals. Thus, by Eqs. (16) and (20),

$$\bar{N}_f = (\bar{x}^T\bar{x})^{-1}\bar{x}^T Ef(x) = (\bar{x}^T\bar{x})^{-1}\bar{x}^T EF(z)$$

$$= [-0.2 \quad -0.2 \quad -0.2]$$

To compute $\tilde{N}_f$ by Eq. (21), expand

$$[R(z-\bar{z})]^T F(z) = [z_1 - 1 \quad 0.6z_1 + 0.8z_2 - 1 \quad z_3 - 1] \times (-0.6z_1^2 - 0.8z_1z_2 + z_3)$$

Note that the high-order moments of the random variables are obtained by using the characteristic function.[9] This gives the optimal equivalent gain for fixed mean and covariance matrix. If such a computation is carried out for a few different values of mean and covariance, the curve-fitting method can then be used to obtain a closed-form approximation to the random-input describing function for the multi-input nonlinearity.

VI. Summary

In this Note, we give a straightforward yet rigorous derivation of the formulas for computing the optimal linear equivalent gains for multi-input/multi-output nonlinearities with random inputs. For general nonlinearities, the computation requires the evaluation of multiple integrals. It is shown that, for some classes of nonlinearities with Gaussian signal inputs, the computation can be greatly simplified and readily carried out numerically. This result has been used in the covariance analysis of interconnected nonlinear systems with multi-input nonlinearities.

References

[1]Zarchan, P., "Efficient Computerized Methods of Statistical Analysis," *Proceedings of the 10th Annual Pittsburgh Conference, Modelling and Simulation, Pt. 2: Systems and Control*, Vol. 10, Pittsburgh, PA, April 25-27, 1979, pp. 577-581.

[2]Gelb, A., and Warren, R. S., "Direct Statistical Analysis of Nonlinear Systems—CADET," AIAA Paper 72-875, Aug. 1972.

[3]Cheng, V. H. L., Curley, R., and Lin, C. A., "A Covariance Analysis Algorithm for Interconnected Systems," *Proceedings of the 1987 AIAA Guidance, Navigation, and Control Conference*, Monterey, CA, Aug. 17-19, 1987.

[4]Gelb, A., and Vander Velde, W. E., *Multiple-Input Describing Functions and Nonlinear System Design*, McGraw-Hill, New York, 1968, Chap. 6.

[5]Atherton, A. P., *Nonlinear Control Engineering*, Van Nostrand Reinhold, New York, 1975, Chap. 8.

[6]Taylor, J. H., "Random-Input Describing Functions for Multi-Input Nonlinearities," *International Journal of Control*, Vol. 23, No. 2, 1976, pp. 277-281.

[7]Stewart, G. W., *Introduction to Matrix Computations*, Academic Press, Orlando, FL, 1973, Chap. 3, p. 134.

[8]Wong, E., *Introduction to Random Process*, Springer-Verlag, New York, 1983, Chap. 2.

[9]Billingsley, P., *Probability and Measure*, Wiley, New York, 1979, Chap. 5.

Constant Covariance in Local Vertical Coordinates for Near-Circular Orbits

Stanley W. Shepperd*

Charles Stark Draper Laboratory, Inc., Cambridge, Massachusetts 02139

Introduction

MANY covariance studies involve near-circular orbits, and it is often desirable to be able to initialize an error covariance matrix that, in the absence of measurements, will remain constant in a rotating local-vertical coordinate system. This Note describes a way to define such a covariance matrix

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*Staff Member, Guidance and Navigation Division. Senior Member AIAA.